JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 3, August 2012

ON CONVERGENCE THEOREMS FOR THE MCSHANE INTEGRAL ON TIME SCALES

Xuexiao You* and Dafang Zhao**

Abstract. In this paper, we study the process of McShane delta integrals on time scales and discuss the relation between McShane delta integral and Henstock delta integral. We also prove the monotone convergence theorem, Fatou's Lemma and the dominated convergence theorems for the McShane delta integral.

1. Introduction

The calculus on time scales was introduced for the first time in 1988 by Hilger [1] to unify the theory of difference equations and the theory of differential equations. It has been extensively studied on various aspects by several authors [2-8]. Surprisingly enough, the McShane integral has not received attention in the literature of time scales. In this paper, a treatment of the McShane integral on time scales is given. We prove the monotone convergence theorem and the dominated convergence theorems for the McShane delta integral. The McShane nabla integral may be treated in a similar way.

2. Definitions and basic properties

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma(t)$ by $\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$ where $\inf \emptyset = \sup\{\mathbb{T}\}$, while the backward jump operator $\rho(t)$ is defined by $\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$ where $\sup \emptyset = \inf\{\mathbb{T}\}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that

Received March 01, 2011; Accepted June 20, 2012.
2010 Mathematics Subject Classification: Primary 26A39; Secondary 26E70.
Key words and phrases: time scales, McShane integral, delta integra.
Correspondence should be addressed to Dafang Zhao, zhaodafang@yahoo.com.

t is left-dense. The forward graininess function $\mu(t)$ of $t \in \mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, while the backward graininess function $\nu(t)$ of $t \in \mathbb{T}$ is defined by $\nu(t) = t - \rho(t)$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Throughout this paper, all considered intervals will be intervals in $\mathbb{T}$. A partition $\mathcal{D}$ of $[a, b]_{\mathbb{T}}$ is a finite collection of interval-point pairs $\{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$, where $\{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ and $\xi_i \in [a, b]_{\mathbb{T}}$ for $i = 1, 2, \cdots, n$. By $\Delta t_i = t_i - t_{i-1}$ we denote the length of the ith subinterval in the partition $\mathcal{D}$. $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$ is a $\Delta-$gauge for $[a, b]_{\mathbb{T}}$ provided $\delta_L(\xi) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(\xi) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0, \delta_R(b) \geq 0$ and $\delta_R(\xi) \geq \mu(\xi)$ for all $\xi \in [a, b)_{\mathbb{T}}$. We say that $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ is

(1) δ - fine *McShane partition* of $[a, b]_{\mathbb{T}}$ if $[t_{i-1}, t_i]_{\mathbb{T}} \subset (\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i))_{\mathbb{T}}$ and $\xi_i \in [a, b]_{\mathbb{T}}$ for all i=1,2,$\cdots$,n,

(2) δ - fine *Henstock partition* of $[a, b]_{\mathbb{T}}$ if it is a δ - fine *McShane partition* of $[a, b]_{\mathbb{T}}$ and satisfying $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}$.

Given a δ - fine *McShane partition* $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ we write

$$S(f, \mathcal{D}) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$$

for McShane $\Delta-$sums over $\mathcal{D}$, whenever $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$.

Definition 2.1. A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is McShane delta integrable (McShane $\Delta-$integrable) on $[a, b]_{\mathbb{T}}$ if there is a number A such that for each $\varepsilon > 0$ there is a $\Delta-$gauge, δ, for $[a, b]_{\mathbb{T}}$ such that

$$|S(f, \mathcal{D}) - A| < \epsilon$$

for each δ- fine *McShane partition* $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ of $[a, b]_{\mathbb{T}}$. A is called the *McShane* $\Delta-$*integral* of f on $[a, b]_{\mathbb{T}}$, and we write $A = \int_a^b f(t)\Delta t$ or $A = (M) \int_a^b f(t)\Delta t$.

Replacing the term McShane partition by Henstock partition in the definition above we obtain Henstock $\Delta-$integrability and the definition of the Henstock $\Delta-$integral $(H) \int_a^b f(t)\Delta t$.

The basic properties of the McShane $\Delta-$integral, for example, linearity and additivity with respect to intervals are similar to the Henstock $\Delta-$integral case. We do not present them here. The reader is referred to [4] for the details.

By the definitions of Henstock $\Delta-$integral and McShane $\Delta-$integral and the fact that each $\delta-$fine Henstock partition is also $\delta-$fine McShane partition, we get immediately the following theorem.

THEOREM 2.2. *If f is McShane $\Delta-$integrable on $[a,b]_{\mathbb{T}}$, then f is Henstock $\Delta-$integrable on $[a,b]_{\mathbb{T}}$.*

REMARK 2.3. The following example shows that the converse of Theorem 2.2 is not true. In other words, there exists a function which is Henstock $\Delta-$integrable but is not McShane $\Delta-$integrable.

EXAMPLE 2.4. *Let $\mathbb{T} = \{t = \frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\}$ and define a function $f : \mathbb{T} \longrightarrow \mathbb{R}$ by*

$$f(t) = \begin{cases} (-1)^n n & \text{if } t = \frac{1}{n}, \\ C & \text{if } t = 0, \end{cases} \tag{2.1}$$

where C is any constant. In [4], Allan Peterson and Bevan Thompson proved that f is Henstock $\Delta-$integrable on $[0,1]_{\mathbb{T}}$ but is not absolutely Henstock $\Delta-$integrable(i.e. both f and $|f|$ are Henstock $\Delta-$integrable). In fact, if f is McShane $\Delta-$integrable on $[0,1]_{\mathbb{T}}$ then so is $|f|$. Further, we also can prove the equivalence of McShane $\Delta-$integrl and absolutely Henstock $\Delta-$integrl. Since the proof is similar to the proof of Theorems 3.12.5 in [9] and hence be omitted. So we have that f is Henstock $\Delta-$integrable but is not McShane $\Delta-$integrable.

LEMMA 2.5. *(Saks-Henstock) Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is McShane $\Delta-$integrable on $[a,b]_{\mathbb{T}}$. Then for each ε there is a $\Delta-$gauge, δ, for $[a,b]_{\mathbb{T}}$ such that*

$$|S(f,\mathcal{D}) - \int_a^b f(t)\Delta t| < \epsilon$$

for each $\delta-$fine McShane partition $\mathcal{D}$ of $[a,b]_{\mathbb{T}}$. Particularly, if $\mathcal{D}' = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^m$ is an arbitrary δ- fine partial McShane partition of $[a,b]_{\mathbb{T}}$, we have

$$|S(f,\mathcal{D}') - \sum_{i=1}^m \int_{t_{i-1}}^{t_i} f(t)\Delta t| \le \epsilon.$$

Proof. The proof is similar to the case for Henstock integrable functions and the reader is referred to [9, Theorem 3.2.1.] for details. □

3. Convergence theorems

LEMMA 3.1. *Let $f_n, f : [a,b]_{\mathbb{T}} \to \mathbb{R}$. Assume that each f_n is McShane $\Delta-$integrable on $[a,b]_{\mathbb{T}}$, $f(t) = \sum_{n=1}^\infty f_n(t)$ pointwise on $[a,b]_{\mathbb{T}}$, and $\sum_{n=1}^\infty \int_a^b |f_n(t)|\Delta t < \infty$. Let $s_k(t) = \sum_{n=1}^k f_n(t)$, then*

(1) *f is McShane $\Delta-$integrable on $[a,b]_{\mathbb{T}}$ with*

$$\int_a^b f(t)\Delta t = \lim_{k\to\infty}\int_a^b s_k(t)\Delta t = \sum_{n=1}^{\infty}\int_a^b f_n(t)\Delta t;$$

(2)

$$\lim_{k\to\infty}\int_a^b |s_k(t)-f(t)|\Delta t = \lim_{k\to\infty}\int_a^b |\sum_{n=k+1}^{\infty} f_n(t)|\Delta t = 0.$$

Proof. Let $\epsilon > 0$ and $A = \sum_{n=1}^{\infty}\int_a^b f_n(t)\Delta t$. Since A is finite, then we can choose $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty}\int_a^b |f_n(t)|\Delta t < \epsilon.$$

For each k, s_k is McShane $\Delta-$integrable and there is a $\Delta-$gauge, δ_k, on $[a,b]_{\mathbb{T}}$ such that

$$|S(s_k,\mathcal{D}_k) - \int_a^b s_k(t)\Delta t| < \frac{\epsilon}{2^k}$$

for each δ_k-fine partition $\mathcal{D}_k$ of $[a,b]_{\mathbb{T}}$.

Define a function $g:\mathbb{R}_{\mathbb{T}}\to\mathbb{R}$ by $g(t) = \frac{1}{4}\sum_{n=1}^{\infty}2^{-n}\chi_{\{t:n-1\le|t|<n\}}$. Then $g(t)$ is McShane $\Delta-$integrable on $\mathbb{R}_{\mathbb{T}}\bigcup\{-\infty,+\infty\}$ and

$$\int_{\mathbb{R}_{\mathbb{T}}\bigcup\{-\infty,+\infty\}} g(t)\Delta t = \frac{1}{2}.$$

Let δ_g be a $\Delta-$gauge such that $|S(g,\mathcal{D}_g) - \int_{\mathbb{R}_{\mathbb{T}}} g(t)\Delta t| < \frac{1}{2}$ for each δ_g-fine partition $\mathcal{D}_g$ of $\mathbb{R}_{\mathbb{T}}$. Further, we have

$$0 \le S(g,\mathcal{D}_g) \le \int_{\mathbb{R}_{\mathbb{T}}} g(t)\Delta t + \frac{1}{2} = 1$$

whenever $\mathcal{D}_g$ is a δ_g-fine partition of $[a,b]_{\mathbb{T}}$.

Since s_k converges pointwise to f, for each $\xi\in[a,b]_{\mathbb{T}}$, we can choose an $k(\xi)\in\mathbb{N}$ such that $k(\xi)\ge N$ and $|s_k(\xi)-f(\xi)| < \epsilon g(\xi)$ for $k \ge k(\xi)$.

Define a $\Delta-$gauge on $[a,b]_{\mathbb{T}}$ by setting $\delta(\xi) = (\delta_L(\xi),\delta_R(\xi))$ such that

$$\delta_L(\xi) = min\{\delta_L^g(\xi),\delta_L^{k(\xi)}(\xi)\}, \delta_R(\xi) = min\{\delta_R^g(\xi),\delta_R^{k(\xi)}(\xi)\}$$

for all $\xi\in[a,b]_{\mathbb{T}}$.

Let $\mathcal{D} = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^m$ be a $\delta-$fine partition of $[a,b]_{\mathbb{T}}$ and

$$M = \max\{k(\xi_1),k(\xi_2),\cdots,k(\xi_m)\} \ge N.$$

Then we have

$$\begin{aligned}
& |S(f,\mathcal{D})-A| \\
= & \ |\sum_{i=1}^{m}[\sum_{n=1}^{\infty} f_n(\xi_i)\Delta t_i - \sum_{n=1}^{\infty}\int_{t_{i-1}}^{t_i} f_n(t)\Delta t]| \\
\leq & \ |\sum_{i=1}^{m}[\sum_{n=1}^{k(\xi_i)} f_n(\xi_i)\Delta t_i - \sum_{n=1}^{k(\xi_i)}\int_{t_{i-1}}^{t_i} f_n(t)\Delta t]| \\
& +|\sum_{i=1}^{m}\sum_{n=k(\xi_i)+1}^{\infty} f_n(\xi_i)\Delta t_i| + |\sum_{i=1}^{m}\sum_{n=k(\xi_i)+1}^{\infty}\int_{t_{i-1}}^{t_i} f_n(t)\Delta t| \\
\leq & \ |\sum_{i=1}^{m}[s_{k(\xi_i)}(\xi_i)\Delta t_i - \int_{t_{i-1}}^{t_i} s_{k(\xi_i)}(t)\Delta t]| \\
& +\sum_{i=1}^{m}|\sum_{n=k(\xi_i)+1}^{\infty} f_n(\xi_i)|\Delta t_i + \sum_{i=1}^{m}\sum_{n=k(\xi_i)+1}^{\infty}\int_{t_{i-1}}^{t_i}|f_n(t)|\Delta t \\
\leq & \ |\sum_{n=N}^{M}\sum_{k(\xi_i)=n}[s_{k(\xi_i)}(\xi_i)\Delta t_i - \int_{t_{i-1}}^{t_i} s_{k(\xi_i)}(t)\Delta t]| \\
& +\sum_{i=1}^{m}|s_{k(\xi_i)}(\xi_i)-f(\xi_i)|\Delta t_i + \sum_{n=N}^{\infty}\int_a^b |f_n(t)|\Delta t \\
\leq & \ \sum_{n=N}^{M}|\sum_{k(\xi_i)=n}[s_{k(\xi_i)}(\xi_i)\Delta t_i - \int_{t_{i-1}}^{t_i} s_{k(\xi_i)}(t)\Delta t]| + \sum_{i=1}^{m}\epsilon g(\xi_i)\Delta t_i + \epsilon \\
\leq & \ \sum_{n=1}^{M}\frac{\epsilon}{2^n} + \epsilon S(g,\mathcal{D}) + \epsilon \\
< & \ 3\epsilon.
\end{aligned}$$

It follows that f is McShane $\Delta-$integrable on $[a,b]_{\mathbb{T}}$ with

$$\int_a^b f(t)\Delta t = \lim_{k\to\infty}\int_a^b s_k(t)\Delta t = \sum_{n=1}^{\infty}\int_a^b f_n(t)\Delta t.$$

Finally, for $k>N$, we have that

$$|\int_a^b s_k(t)\Delta t \quad - \quad \int_a^b f(t)\Delta t| \leq \int_a^b |s_k(t)-f(t)|\Delta t$$

$$= \int_a^b |\sum_{n=k+1}^{\infty} f_n(t)|\Delta t \leq \sum_{n=N}^{\infty} \int_a^b |f_n(t)|\Delta t < \epsilon$$

then $\int_a^b f(t)\Delta t = \lim_{k\to\infty} \int_a^b s_k(t)\Delta t$ and $\int_a^b |s_k(t) - f(t)|\Delta t \to 0$, completing the proof. □

REMARK 3.2. The Lemma 3.1 also holds if the interval $[a, b]_{\mathbb{T}}$ is unbounded.

THEOREM 3.3. *(Monotone Convergence Theorem). Let $f_n, f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ and assume that*

(i) $f_n \leq f_{n+1}$ on $[a, b]_{\mathbb{T}}$;

(ii) f_n is McShane $\Delta-$integrable and $\sup_n \int_a^b f_n(t)\Delta t < \infty$, $n \in \mathbb{N}$;

(iii) $f_n \to f$ in $[a, b]_{\mathbb{T}}$.

Then f is McShane $\Delta-$integrable on $[a, b]_{\mathbb{T}}$ and

$$\int_a^b f(t)\Delta t = \int_a^b \lim_{n\to\infty} f_n(t)\Delta t = \lim_{n\to\infty} \int_a^b f_n(t)\Delta t.$$

Proof. Set $f_0 = 0$ and $g_n = f_n - f_{n-1}$ for $n \geq 1$. Then $g_n \geq 0$ and

$$\lim_{n\to\infty} \sum_{k=1}^{n} g_k = \lim_{n\to\infty} f_n = f.$$

So we have that

$$\begin{aligned} \sum_{n=1}^{\infty} \int_a^b g_n(t)\Delta t &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b (f_k(t) - f_{k-1}(t))\Delta t \\ &= \lim_{n\to\infty} \int_a^b f_n(t)\Delta t = \sup_n \int_a^b f_n(t)\Delta t < \infty. \end{aligned}$$

Consequently, from Lemma 3.1 we have

$$\int_a^b f(t)\Delta t = \int_a^b \lim_{n\to\infty} f_n(t)\Delta t = \sum_{n=1}^{\infty} \int_a^b g_n(t)\Delta t = \lim_{n\to\infty} \int_a^b f_n(t)\Delta t.$$

□

LEMMA 3.4. *(Fatou's Lemma) Let $f_n, g : [a, b]_{\mathbb{T}} \to \mathbb{R}^+$ be McShane $\Delta-$integrable for $n = 1, 2 \cdots$. Assume that $f_n \geq g$ and*

$$\liminf_{n\to\infty} \int_a^b f_n(t)\Delta t < \infty.$$

Then $f = \liminf_{n\to\infty} f_n$ *is McShane* $\Delta-$*integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$\int_a^b f(t)\Delta t = \int_a^b \liminf_{n\to\infty} f_n(t)\Delta t \leq \liminf_{n\to\infty} \int_a^b f_n(t)\Delta t.$$

Proof. Let $g_n = \inf_{k\geq n} f_k$ for $n = 1, 2, \cdots$, then g_n is McShane $\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and g_n increases monotonically to f. Since $g_n \leq f_n$ for all n, we have

$$\int_a^b g_1(t)\Delta t \leq \lim_{n\to\infty} \int_a^b g_n(t)\Delta t \leq \liminf_{n\to\infty} \int_a^b f_n(t)\Delta t.$$

From Theorem 3.3 we have that f is McShane $\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$\int_a^b f(t)\Delta t = \lim_{n\to\infty} \int_a^b g_n(t)\Delta t.$$

□

Theorem 3.5. *(Dominated Convergence Theorem) Let* $f_n, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *be McShane* $\Delta-$*integrable and suppose that* $|f_n| \leq g$ *for* $n = 1, 2 \cdots$. *If* $f = \lim_{n\to\infty} f_n$, *then* f *is McShane* $\Delta-$*integrable and*

$$\int_a^b f(t)\Delta t = \int_a^b \lim_{n\to\infty} f_n(t)\Delta t = \lim_{n\to\infty} \int_a^b f_n(t)\Delta t.$$

Proof. From Lemma 3.4 we have that f is McShane $\Delta-$integrable on $[a,b]_{\mathbb{T}}$. Applying Fatou's Lemma to the sequences $\{f_n\}$ and $\{-f_n\}$, we have that

$$\begin{aligned} \int_a^b f(t)\Delta t &= \int_a^b \liminf_{n\to\infty} f_n(t)\Delta t \leq \liminf_{n\to\infty} \int_a^b f_n(t)\Delta t \\ &\leq \limsup_{n\to\infty} \int_a^b f_n(t)\Delta t \leq \int_a^b \limsup_{n\to\infty} f_n(t)\Delta t = \int_a^b f(t)\Delta t \end{aligned}$$

and the result follows. □

Corollary 3.6. *Let* $f_n, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *be McShane* $\Delta-$*integrable and suppose that* $|f_n(t)| \leq g(t)$ *a.e. for* $t \in [a,b]_{\mathbb{T}}$. *If* $f_n \to f$ *a.e. in* $[a,b]_{\mathbb{T}}$, *then* f *is McShane* $\Delta-$*integrable and*

$$\lim_{n\to\infty} \int_a^b |f_n(t) - f(t)|\Delta t = 0.$$

Acknowledgement

The authors are grateful to the referee for his or her careful reading of the manuscript and for valuable and helpful suggestions.

References

[1] S. Hilger, *Ein Makettenkalkl mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph. D. Thesis, Universtat Wurzburg, 1988.

[2] S. Hilger, *Analysis on measure chainsA unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18-56.

[3] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhauser, Boston, 2004.

[4] A. Peterson, B. Thompson, *HenstockCKurzweil Delta and Nabla Integrals*, J. Math. Anal. Appl. **323** (2006), 162-178.

[5] G. Sh. Guseinov, *Integration on time scales*, J. Math. Anal. Appl. **285** (2003), 107-127.

[6] G. Sh. Guseinov, B. Kaymakcalan, *Basics of Riemann delta and nabla integration on time scales*, J. Difference Equations Appl. **8** (2002), 1001-1027.

[7] S. Avsec, B. Bannish, B. Johnson, and S. Meckler, *The Henstock-Kurzweil delta integral on unbounded time scales*, PanAmerican Math. J. **16** (2006), no. 3, 77-98.

[8] B. S. Thomson, *Henstock-Kurzweil integrals on time scales*, PanAmerican Math. J. **18** (2008), no. 1, 1-19.

[9] L. P. Yee and R. Vyborny, *The integral, An Easy Approach after Kurzweil and Henstock*, Australian Mathematical Society Lecture Series 14, Cambridge University Press, 2000.

[10] C. W. Swartz, Douglas S Kurtz, *Theories of Integration: The Integrals of Riemann, Lebesgue, Henstock-Kurzweil, and Mcshane*, World Scientific, 2004.

[11] D. Zhao and G. Ye, *C-integral and Denjoy-C integral*, Comm. Korean. Math. Soc. **22** (2007), no. 1, 27-39.

[12] D. Zhao and G. Ye, *On AP-Henstock-Stieltjes integral*, J. Chungcheong Math. Soc. **19** (2006), no. 2, 177-188.

[13] D. Zhao and G. Ye, *On strong C-integral of Banach-valued functions*, J. Chungcheong Math. Soc. **20** (2007), no. 1, 1-10.

[14] J. M. Park, *The Denjoy extension of the McShane integral*, Bull. Korean Math. Soc. **33** (1996), no. 3, 411-416.

*

School of Mathematics and Statistics
Hubei Normal University
Huangshi, 435002, People's Republic of China
E-mail: youxuexiao@126.com

**

School of Mathematics and Statistics
Hubei Normal University
Huangshi, 435002, People's Republic of China
E-mail: zhaodafang@yahoo.com